Proceedings of the Seventh Fuel and Combustion Conference of Iran
Feb. 13 - 14, 2018, Sharif University of Technology, Tehran, Iran
FCCI-2018-1086

Semi-analytical solution of transient imperfectly expanded turbulent supersonic jet

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Abstract

Although the steady state flow has been taken into the account widely, studying the transient flow is not a common investigation through the published analytical researches. In the present work, deriving an analytical solution, the transient behavior of imperfectly expanded supersonic jet flow has been studied in detail. In this regard applying Laplace transform method, the Favre-averaged Navier-Stokes equations for turbulent compressible flow have been solved. Then the results are validated with the available experimental data in the literature. Furthermore, the effects of eddy viscosity on steady state flow is discussed. It is found that velocity increases by decreasing the eddy viscosity. Moreover, tip penetration, radial velocity and velocity profile are studied and plotted. As expected results show that pressure ratio has direct relationship with tip penetration length. So, in can be concluded that the present analytical solution can have profound impact on developing future studies.

Keywords: imperfectly expanded jet, semi-analytical solution, transient flow.

INTRODUCTION

Studying of imperfectly expanded jets is a key topic in many engineering applications, such as: screech noise reduction in aircraft, design of aeronautic vehicles, etc.. The pressure of the flow field is decreased/increased, when a multi-cell shock structure is formed. This formation is because of the higher/lower pressure of supersonic jet at a nozzle exit than the ambient pressure. As it is obvious in Figure 1 [1], despite a simple flow geometry in imperfectly expanded free jets, they are very intricate phenomena due to shock waves and decay of shock cells by getting away from the nozzle exit [2].

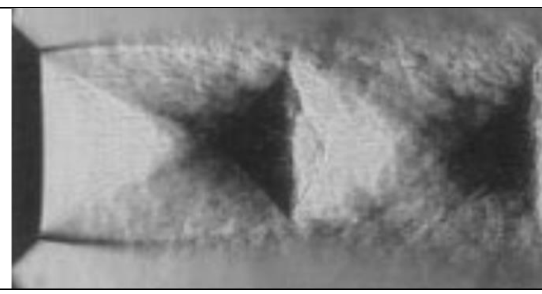

Figure 1- Schlieren image of an under expanded jet at Mach number of 1.42 [1].

Even though several numerical [3-6] and experimental [7-13] studies in jets have been conveyed, the interpretation of the physical aspects of phenomena is the preference of analytical methods [14-16].

Pack [17] linearized and solved the velocity potential equation in compressible inviscid flow. Kleinstein [18] demonstrated an analysis to find a solution for mixing problems entailing mass transfer in turbulent axially symmetric compressible flow, and momentum energy. Linearizing the governing equations in the plane of the von Mises variables, the aforementioned technique was developed and formed. Also, results of laminar flow problem in his research are in a good agreement with presented numerical results in Pai's article [19]. The properties of spray penetration regarding to the momentum theory, and its basic idea are discussed in the Wakuri et al.'s article [20]. The idea is that the air induced into a fuel jet stream leads to fuel droplets and mixed gas. Sifer [21] investigates the air jet which is discharging from rectangular slots and channels in stationary atmosphere. It is explained that in 50% velocity spread, the jet growth rate was greater in minor axis direction, compared to others. Moreover, he reported the experimental results of turbulence shear stress profile, lateral and longitudinal turbulence intensity, and various axes of mean velocity. He also compared these results with those which are gained for two dimensional jets. A theoretical model that outlines the creation of sound by an imperfectly expanded supersonic jet is explained in Howe and Ffowcs-Williams' article [22]. Their results indicated that in the presence of the system of shock cells, the total sound power is in the range of 0.2 and 1.2% of the exhaust power of a supersonic transport jet in take-off condition. They also validated their results with the experimental results. The screech noise in self-exciting steady jets has been analytically and experimentally examined in Panda's [1] research. A new method for shock detection along the Schlieren photography are the base of performed experiments. Analytical method has been performed to clarify the jet column self-excitation via the screech sound. A semi experimental method was used by Hill and Ouelette [23] to predict the under expanded gas penetration. The effects of injection duration, pressure ratio, wall constraints, and lift on the jet are determined semi analytically and experimentally. A good agreement between their model and experimental data is validated and shown. Emami et al. [2] revealed that in the most of previous researches, the solutions were derived by neglecting turbulent and molecular diffusions in the inviscid equations. Hence, considering the mean Reynolds stresses, they reported an analytical solution with the aid of extending the linearized solution of the Navier-Stokes equations. In this regard, the value of eddy viscosity was determined employing the empirical turbulence model of Witz [24]. Because of linearization error, the magnitude of pressures achieved in their work has some differences with the experiments, however their results are in a good agreement with the available experiments [25, 26]. An analytical solution for modeling the unsteady inviscid jet is employed by De Chant [14], taking into account the velocity potential theory. A good compatibility has been demonstrated in his work by comparing jet shape with a numerical one. The transient under expanded jet flow has not been investigated deeply by analytical methods, since transient set of equations are too complex. Chitsaz et al. [27] studied the transient under expanded jet flow has been solved semi-analytically applying Hankel transform. Although the equation of pressure was solved by them, there is no proposal solution for velocity field. They clarified that the best process for steady state under expanded jet flow is the isentropic one.

In this study, the transient imperfectly expanded supersonic turbulent jet flow is examined and simplified regarding dimensional analysis. Moreover, an analytical solution for the transient imperfectly expanded supersonic jet flow is described, using the numerical inverse Laplace transform. Besides, comparison between existing experimental data in the literature and the results of this study is presented which shows good agreement. Furthermore, the transient imperfectly expanded supersonic turbulent jet flow equations are obtained. In addition, the results of steady and unsteady conditions are validated with the former experimental data, and the theoretical aspects of them are described as well. Effect of various parameters on the jet structure like pressure ratio and eddy viscosity is also examined.

MATHEMATICAL MODEL

Neglecting the molecular viscous terms, the Favre-averaged Navier-Stokes equations for turbulent compressible flow can describe the phenomena. So, continuity and momentum equations for turbulent compressible flow are displayed as follow,

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla.(\bar{\rho}\tilde{u}) = 0 \tag{1}$$

$$\bar{\rho}\frac{\partial \tilde{u}}{\partial t} + \bar{\rho}u\nabla.\tilde{u} = -\nabla\bar{P} + \mu_t\nabla^2\tilde{u} + \frac{\mu_t}{3}\nabla(\nabla.\tilde{u}) \tag{2}$$

Eddy viscosity is approximately constant in the high speed turbulent jets. This assumption is based on the Kleinstein [18] and Witze [24] suggested empirical correlations. In this investigation, the Kleinstein suggested equation [18] has been used as,

$$\mu_t = 0.00915(\rho_e\rho_\infty)^{0.5}U_eD_e \tag{3}$$

Since Chitsaz et al. [27] showed that the order of magnitude of radial and axial derivatives of pressure are the same, the following dimensional analysis is considered,

$$\frac{\frac{\partial \bar{P}}{\partial x}}{\frac{\partial \bar{P}}{\partial r}} \approx O(1) \rightarrow \frac{\frac{P_e - P_\infty}{\delta x}}{\frac{P_e - P_\infty}{\frac{D_e}{2}}} \approx O(1) \rightarrow \frac{D_e}{2\delta x} \approx O(1) \rightarrow \frac{\frac{\partial \tilde{u}}{\partial x}}{\frac{\partial \tilde{v}}{\partial r}} \approx \frac{\frac{U_e}{\delta x}}{\frac{V}{\frac{D_e}{2}}} \approx \frac{U_e}{V} \gg 1 \quad \& \quad \frac{\tilde{u}\frac{\partial \bar{\rho}}{\partial x}}{\tilde{v}\frac{\partial \bar{\rho}}{\partial r}} \approx \frac{\frac{U_e(\rho_e - \rho_\infty)}{\delta x}}{\frac{V(\rho_e - \rho_\infty)}{\frac{D_e}{2}}} \approx \frac{U_e}{V} \gg 1 \tag{4}$$

It can be concluded from Eq. (4) that the radial component of velocity is negligible compared to the axial one. With this in mind and considering the above-mentioned dimensional analysis, radial derivative of density is neglected compared with axial one.

Following equations for an axisymmetric imperfectly expanded turbulent jet are concluded by means of linearizing the continuity and momentum equations applying ρ_e and U_e,

$$\frac{\partial\bar{\rho}}{\partial t}+\rho_e\frac{\partial\tilde{u}}{\partial x}+U_e\frac{\partial\bar{\rho}}{\partial x}=0 \tag{5}$$

$$\rho_e\frac{\partial^2\tilde{u}}{\partial t\partial r}+\rho_e U_e\frac{\partial^2\tilde{u}}{\partial x\partial r}=\mu_t\left[\frac{\partial^3\tilde{u}}{\partial x^2\partial r}+\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{u}}{\partial r}\right)\right\}\right] \tag{6}$$

$$\tilde{u}(x,r,t)\overset{L_t}{\rightarrow}T(x,r,s) \tag{7}$$

Considering Eq. (7) and applying Laplace transform to time, the momentum equation will be as follows,

$$\rho_e\frac{\partial}{\partial r}[sT(x,r,s)-u_x(x,r,t=0)]+\rho_e U_e\frac{\partial^2 T}{\partial r\partial x}=\mu_t\left[\frac{\partial^3 T}{\partial x^2\partial r}+\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] \tag{8}$$

$$T(x,r,s)\overset{L_x}{\rightarrow}X(p,r,s) \tag{9}$$

Regarding Eq. (9) and introducing Laplace transform to axial direction, the momentum equation (8) will be simplified as follows,

$$\rho_e\frac{\partial}{\partial r}(sX(p,r,s))+\rho_e U_e\frac{\partial}{\partial r}[PX(p,r,s)-T(x=0,r,s)]=\mu_t\left[\frac{\partial}{\partial r}(P^2X(p,r,s))-\frac{\partial}{\partial r}(PT(x=0,r,s)-\frac{\partial^2 T}{\partial x\partial r}(x=0,r,s)+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial X}{\partial r}\right)\right)\right] \tag{10}$$

The following equation shows the boundary conditions at transformed space,

$$T(x=0,r,s)=\frac{U_e}{s}, \qquad \frac{\partial T}{\partial x}(x=0,r,s)=0 \tag{11}$$

Substituting Eq. (11) in Eq. (10) leads to the following equation,

$$\rho_e\frac{\partial}{\partial r}(sX)+\rho_e U_e\frac{\partial}{\partial r}\left[PX-\frac{U_e}{s}\right]=\mu_t\left[\frac{\partial}{\partial r}(P^2X-P\frac{U_e}{s})+\frac{\partial^3 X}{\partial r^3}+\frac{1}{r}\frac{\partial^2 X}{\partial r^2}-\frac{1}{r^2}\frac{\partial X}{\partial r}\right] \tag{12}$$

The final solution of Eq. (12) is as follows,

$$X(p,r,s)=-\frac{iC_1J_0\left(ir\sqrt{\frac{\rho_e s}{\mu_t}+\left(\frac{\rho_e U_e}{\mu_t}-p\right)p}\right)}{\sqrt{\frac{\rho_e s}{\mu_t}+\left(\frac{\rho_e U_e}{\mu_t}-p\right)p}}-\frac{iC_2Y_0\left(-ir\sqrt{\frac{\rho_e s}{\mu_t}+\left(\frac{\rho_e U_e}{\mu_t}-p\right)p}\right)}{\sqrt{\frac{\rho_e s}{\mu_t}+\left(\frac{\rho_e U_e}{\mu_t}-p\right)p}}+\frac{iC_1}{\sqrt{\frac{\rho_e s}{\mu_t}+\left(\frac{\rho_e U_e}{\mu_t}-p\right)p}}+C_3 \tag{13}$$

Equation (13) should be mapped from $X(p,r,s)$ function space to the $\tilde{u}(x,r,t)$ space, considering that the velocity magnitude must be finite. Due to complexity of the above-mentioned equation, numerical inverse Laplace transform has been performed.

RESULTS AND DISCCUSION

The following equations are used, to non-dimensionalize the different quantities,

$$u^*=\frac{u}{U_e}\ ,\quad x^*=\frac{x}{D_e}\ ,\quad r^*=\frac{r}{D_e}\ ,\ P^*=\frac{P}{P_\infty}\ ,\quad t^*=\frac{tU_e}{D_e} \tag{14}$$

As it is obvious in Figure 2, the non-dimensionalized axial velocity of centerline is compared with available experimental data reported by Lau [28]. In that report, Mach number was 1.37 at the exit of nozzle, and it is clearly demonstrated that the present solution can predict the experimental results well.

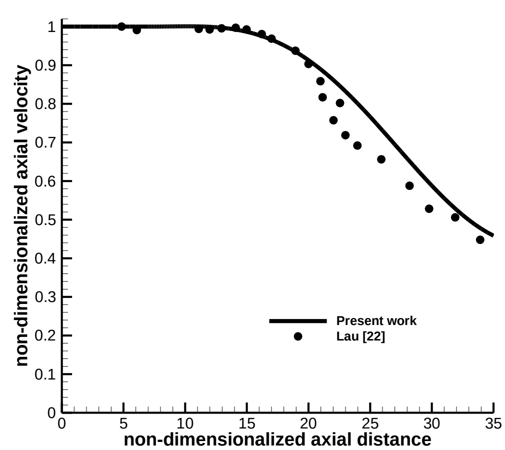

Figure 2. Axial velocity of centerline versus axial distance.

Panda and Seasholtz [9] studied the compressible turbulent jet flow. Their experimental tests are used to perform another validation. Figure 3 shows the comparison between the results of the present solution and experimental jet flows of Panda and Seasholtz [9] at Mach number of 1.4. This figure indicates that the present analytical model is validated well with experiment.

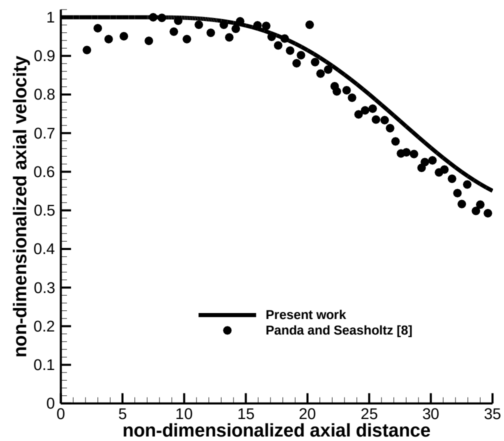

Figure 3. Centerline velocity with exit Mach of 1.4 versus axial distance.

The near-field velocity fluctuations are described in panda and Seaholtz's article [9] as a result of shock occurrence. The achieved mean velocity from experiment is adjacent to the predicted current solution; but, because of simplification and linearization, the present method cannot model shock behavior.

Eddy viscosity, μ_t, is one of the key parameters which affects the nature of flow and is considered in our formulations. Eddy viscosity alters the behavior of jet in a way that the more eddy viscosity, the faster damping. The variation of centerline velocity by axial distance for different eddy viscosities are demonstrated in Figure 4. As expected, increasing the eddy viscosity increases the rate of decreasing velocity along the axial distance sharply.

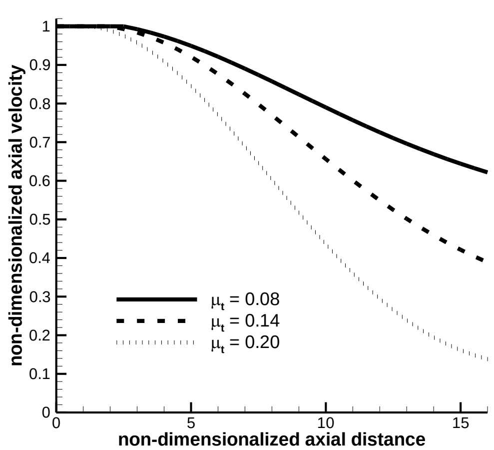

Figure 4. Centerline velocity vs. axial distance at different turbulent eddy viscosities.

It is noteworthy to study the behavior of flow in different radial and axial positions. In this regard, the velocity profile versus axial distance for different radial positions has been plotted and shown in Figure 5. As it is visible from this figure, the more distance from centerline, the less axial velocity.

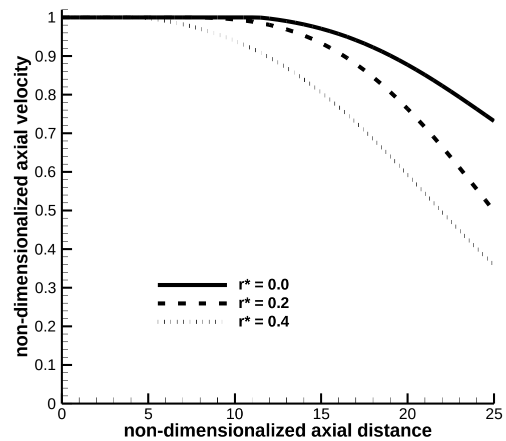

Figure 5. Variation of axial velocity versus axial distances at different radial positions.

Tip penetration versus time is one of the most popular factors which is investigated to explain the behavior of the unsteady jet flow. Figure 6 shows the non-dimensionalized tip penetration versus non-dimensionalized time which is obtained from present solution.

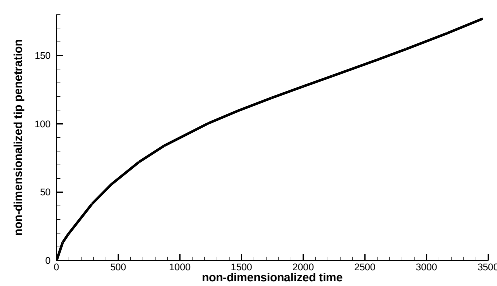

Figure 6. Tip penetration length versus time.

The effects of pressure ratio on tip penetration length for three different pressure ratios is plotted in Figure 7. In this figure, it is assumed that temperature at nozzle exit is constant. It is notable to mention that the increase of pressure causes the increase in momentum flow. So, as pressure increases at the nozzle exit, the jet flow go further and subsequently provides the higher tip penetration length.

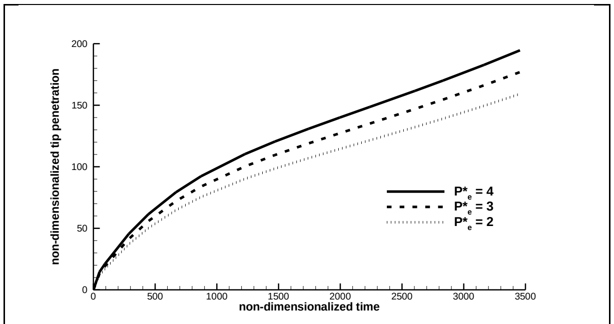

Figure 7 Tip penetration length versus time for different pressure ratios.

Figure 8 shows the velocity profiles at various distances from the nozzle exit. The whole behavior of flow is displayed in this picture and the different behaviors of flow in axial and radial coordinates can be examined. As this figure shows, the shape of velocity profile from the near field to the far field changed from "top-hat" to "Gaussian", respectively.

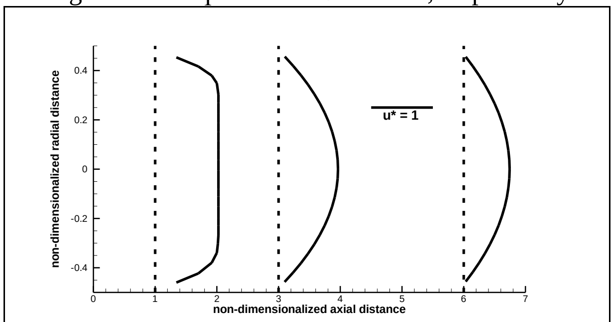

Figure 8. Profiles of velocity at different locations.

CONCLUSION

In this study, a semi-analytical method for imperfectly expanded turbulent jet is performed, based on Favre-averaged Navier-Stocks equations. An equation as a function of two axial and radial spatial coordinates and time is obtained for velocity of turbulent imperfectly expanded free jet. The aforementioned equation is mapped based on Laplace transform method to resolve the controlling equations. Explaining the velocity progress and tip penetration for various times in the unsteady part is one of the main privileges of the present research. A comparison between some available results in literature and the described solution has been performed, and they approve each other.

Afterward, the effects of eddy viscosity and pressure ratio on jet flow behaviors have been surveyed. Moreover, plotting the velocity profiles for different locations are done to achieve a general behavior of flow. Using the introduced method and formulation can be helpful to understand the physical features of imperfectly-expanded jet flows from the beginning of injection to the entirely matured shape.

Acknowledgement

The present study has been performed under support of National Elites Foundation.

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